Forbidden Sets in Argumentation Semantics

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Overview

Forbidden Sets vs. Extension membership

Basic Properties of Forbidden Sets

Forbidden sets and relationship with divers semantics.

Computational Complexity

Further questions.

Forbidden Sets I

In Dung's formalism, given an AF – $< \mathcal{X}, \mathcal{A} >$ – groups of "collectively acceptable" arguments are often defined as those subsets of $\mathcal{X}$ satisfying various criteria.

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conflict free.

$\subseteq$-maximally conflict-free (so-called *naive* extensions).

conflict-free and *defensive* (so-called *admissible* sets)

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S belonging to such systems is characterized "positively".

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An argument in S attacks another argument in S.

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There is an unattacked outsider.

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Such T are the **forbidden sets** (for the relevant semantics)

Forbidden Sets III - Formal definition

For finite set $\mathcal{X}$, let $\mathbb{S}$ be a subset of $2^{\mathcal{X}}$, ie $\mathbb{S}$ is a set of subsets of $\mathcal{X}$.
The **forbidden sets** for $\mathbb{S}$ are

$$\kappa(\mathbb{S}) \quad = \quad \{\ T \subseteq \mathcal{X}\ :\ \forall S \in \mathbb{S}\ \ \neg(T \subseteq S)\ \}$$

The **minimal** forbidden sets for $\mathbb{S}$ are

$$\mu(\mathbb{S}) \quad = \quad \{\ Q\ :\ Q \in \kappa(\mathbb{S})\ \textbf{and}\ \ (R \subset Q) \Rightarrow (R \notin \kappa(\mathbb{S})\}$$

Some basic properties of forbidden sets

For $\mathbb{S} \subseteq 2^{\mathcal{X}}$,

a. For every $Q \in \kappa(\mathbb{S})$ there is some $R \subseteq Q$ with $R \in \mu(\mathbb{S})$.
b. $(\emptyset \in \kappa(\mathbb{S})) \ \equiv \ (\mu(\mathbb{S}) = \{\emptyset\}) \ \equiv \ (\mathbb{S} = \emptyset)$.
c. $\kappa(\mathbb{S}) = \emptyset$ if and only if $\{x_1, x_2, \ldots, x_n\} \in \mathbb{S}$.

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c. $\kappa(\mathbb{S}) = \emptyset$ if and only if $\{x_1, x_2, \ldots, x_n\} \in \mathbb{S}$.

Informal statement

a. If Q is a forbidden set for $\mathbb{S}$ then some subset, R of Q is a minimal such set.

b. The empty set is a forbidden set for $\mathbb{S}$ if and only if $\mathbb{S}$ itself is empty.

c. $\mathbb{S}$ has no forbidden set if and only if $\mathbb{S}$ contains the set $\mathcal{X}$.

Forbidden sets and extension-based semantics

Many results on extension-based semantics in AFs are concerned with the relationships between extension sets of distinct semantics, σ and τ.

Properties of Forbidden sets deriving from Extension properties

a. If σ and τ satisfy "every σ-extension is a τ-extension" then "every forbidden set for τ-extensions is also a forbidden set for σ-extensions".

b. If the σ-extensions are the $\subseteq$-maximal τ-extensions, then their forbidden sets coincide.

c. For σ, τ as in (b), a set S is a σ-extension iff,

   c1. S contains a forbidden set for $\mathcal{E}_\tau \setminus \mathcal{E}_\sigma$. **AND**

   c2. S does *not* contain a forbidden set for $\mathcal{E}_\sigma$.

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The condition (c2) alone is not enough to distinguish $S \in \mathcal{E}_\sigma$ on account of (b).

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b. The conflict-free and naive semantics have the same systems of forbidden sets.

c. The forbidden sets for $\mathcal{E}_{pr}$ are a subset of those for $\mathcal{E}_{sst}$ which in turn are a subset of those for $\mathcal{E}_{st}$.

d. The forbidden sets for conflict-free semantics are a subset of those for admissibility semantics.

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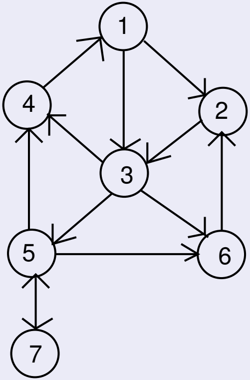
The relationships from (c) and (d) are tight.

Example, $\mu(\mathcal{E}_{pr}) \not\subseteq \mu(\mathcal{E}_{st})$



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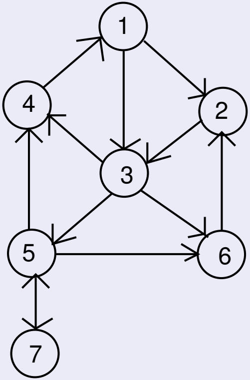

$$
\begin{array}{lcl}
\mathcal{E}_{pr} & = & \{\{1,5\},\ \{7\}\} \\
\mathcal{E}_{st} & = & \{\{1,5\}\} \\
\mu(\mathcal{E}_{pr}) & = & \{\{2\},\ \{3\},\ \{4\},\ \{6\},\ \{1,5,7\}\} \\
\mu(\mathcal{E}_{st}) & = & \{\{2\},\ \{3\},\ \{4\},\ \{6\},\ \{7\}\}
\end{array}
$$

Characterizations

Conflict-free & Naive semantics

$$\mu(\mathcal{E}_{cf}) \;=\; \{\ \{x, y\}\ :\ < x, y > \in \mathcal{A}\ \}$$

Admissibility

$$\mu(\mathcal{E}_{adm}) \;=\; \min_{\subseteq}\ \{\ S\ :\ S \text{ is defenceless}\ \}$$

That is, every conflict-free superset of S has an attacker which is not defended.

Unique status semantics

$$\mu(\mathcal{E}_{sing}) \;=\; \{\ \{x\}\ :\ x \notin E_{sing}\ \}$$

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For any AF, $< \mathcal{X}, \mathcal{A} >$ with $|\mathcal{X}| = n$, no set $S \in \mu(\mathcal{E}_{pr})$ can contain more than $n - \log_2 n$ members.

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Upper bound

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Lower bound

There are AFs with $S \in \mu(E_{pr})$ having $\lfloor n/2 \rfloor$ elements.

Complexity of minimal forbidden set recognition

Single argument sets

With the exception of **stable** semantics, deciding if $\{x\} \in \mu(\mathcal{E}_\sigma)$ is simply a reformulation of the non-credulous acceptance problem, and thus its complexity is the complementary class of credulous acceptance.

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Two argument sets - admissibility semantics

The question $\{x, y\} \in \mu(\mathcal{E}_{pr})$ turns out to be more challenging. To verify such cases requires showing:

1. That $\{x, y\}$ **is** a forbidden set. **AND**
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The second involves testing that both x and y are credulously accepted; the first that the two are not simultaneously so.

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This structure leads to a straightforward D^p membership algorithm which is "optimal": the decision problem is also D^p-hard.

Further Development

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1. $S \in \mu(\mathcal{E}_\sigma)$ with $|S| = 2$ corresponds to "conflict-sensitivity" in recent work of Dvorak *et al.*. What do $S \in \mu(\mathcal{E}_\sigma)$ having $|S| \geq 3$ tell us about realizability issues?

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2. Since $\mu(\mathbb{S})$ is itself a subset of $2^{\mathcal{X}}$, in principle the operator can be iterated. How does $\mu^k(\mathcal{E}_\sigma)$ relate to $\mathcal{E}_\sigma$?