Spectral Techniques in Argumentation Framework Analysis

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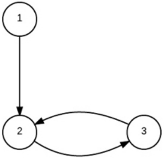
Overview

Introduction

- The purpose of this study was to consider the feasibility of using *spectral techniques* to analyse Dung's abstract argumentation frameworks introduced in Dung (1995).
- Spectral Graph Theory - The study of properties of a graph in relation to the *eigenvalues* and *eigenvectors* of associated matrices of the graph (in our case, the adjacency matrix).

Introduction

Spectral Graph Theory



Graph

$$\begin{array}{cc} \mathit{Vertex} & \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \end{array}$$

Adjacency Matrix

$$\lambda_1 = 1$$
$$\lambda_2 = -1$$
$$\lambda_3 = 0$$

Eigenvalues

Motivation

Spectral techniques offer important insights into many different areas of science, where the problem at hand can be represented as a graph. Most notably:-

- Google's PageRank algorithm, see Bryan and Leise (2006),
- Characterisation of human faces, see Kirby and Sirovich (1990),

Motivation

Spectral techniques offer important insights into many different areas of science, where the problem at hand can be represented as a graph. Most notably:-

- Analysis of molecular structure, see Estrada (2000),

Image from Estrada 2000

- Subgraph centrality in complex networks, see Estrada and Rodriguez-Velazquez (2005).

Overview

Dung's Abstract Argumentation Frameworks

Definition 1

An argumentation framework (AF) is a pair $\mathcal{H} = (\mathcal{X}, \mathcal{A})$.

$\mathcal{X}$ is a finite set of arguments.

$\mathcal{A}$ is a binary relation on $\mathcal{X}$ such that $\mathcal{A} \subset \mathcal{X} \times \mathcal{X}$, known as the *'attacks'* relation.

Dung's Abstract Argumentation Frameworks

Definition 2

Let $\mathcal{S}$ be an arbitrary subset of arguments of $\mathcal{X}$.

$\mathcal{S}$ is *conflict-free* iff no argument in $\mathcal{S}$ is attacked by any other argument in $\mathcal{S}$.

The set $\mathcal{S}$ is a *stable extension* iff it is conflict-free and $\mathcal{S}$ attacks each argument that does not belong to $\mathcal{S}$.

Matrix Algebra

Definition 3

Let $\mathbf{M}^{\mathcal{H}}$ be the n x n (0,1)-matrix representing $\mathcal{H}$, where $n = |\mathcal{X}|$.

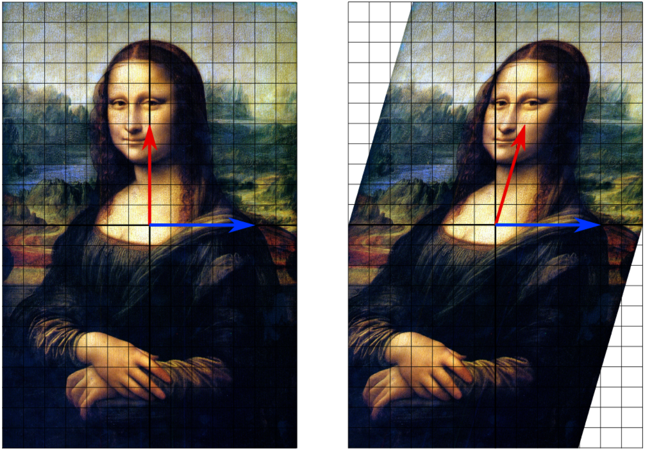
Entry $m_{ij} = 1$ iff $\langle x_i, x_j \rangle \in A$, otherwise $m_{ij} = 0$.

Definition 4

Let λ be a complex number, λ is an *eigenvalue* of $\mathbf{M}^{\mathcal{H}}$ iff there exists some non-zero, n x 1 vector $\underline{v}$ such that $\mathbf{M}^{\mathcal{H}}\underline{v} = \lambda\underline{v}$.

$\underline{v}$ is an *eigenvector* with respect to $\mathbf{M}^{\mathcal{H}}$ and λ in this case.

Matrix Algebra



TreyGreer62, *Mona Lisa Eigenvector Grid*, Retrieved from:
https://en.wikipedia.org/wiki/Eigenvalues_and_eigenvectors

Matrix Algebra

Definition 5

The tuple

$$\sigma(\mathcal{H}) = \langle \lambda_1, \lambda_2, ..., \lambda_n \rangle$$

formed by the n eigenvalues of $\mathbf{M}^{\mathcal{H}}$ is called the *spectrum* of $\mathcal{H}$.

An ordering of $\sigma(\mathcal{H})$ is assumed such that whenever $i \leq j$, it holds that $|\lambda_i| - |\lambda_j| \geq 0$. The eigenvalues are considered in a *non-decreasing* order and λ_1 is the largest eigenvalue (dominant).

Where for,

$$\lambda = a + ib \in \mathbb{C}, \quad \text{where, } i = \sqrt{-1}$$

$$|\lambda| = \sqrt{(a^2 + b^2)}$$

Matrix Algebra

Definition 6

The *Estrada Index*, **EE**, of $\mathcal{H}$ - defined in Estrada (2000) - is given as,

$$\mathbf{EE}(\mathcal{H}) = \sum_{\lambda \in \sigma(\mathcal{H})} e^{\lambda}$$

Fact 1

For all $\mathcal{H}$, $\mathbf{EE}(\mathcal{H}) \in \mathbb{R}$.

Overview

Experimental Aim

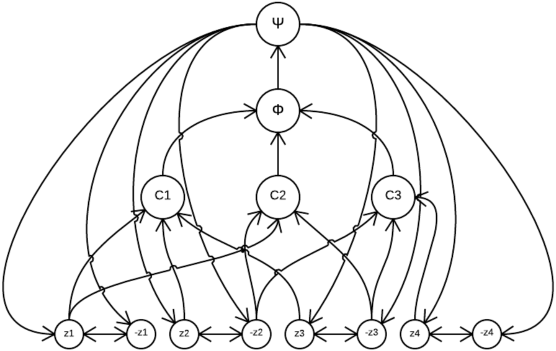
Aim

(Somewhat) randomly generate AFs, calculate their respective spectra and assess whether there are any links between the spectra and a specific semantic property of the AF (in our case the existence of a stable extension).

Not **all** AFs are considered, only a specific set.

We construct AFs from 3-CNF formula; specifically in the form given by Dimopoulos and Torres (1996).

Considered Frameworks



$$\phi = (z_1 \vee z_2 \vee z_3) \wedge (z_1 \vee \neg z_2 \vee \neg z_3) \wedge (\neg z_2 \vee \neg z_3 \vee z_4)$$

Considered Frameworks

Fact 2

Let ϕ be any CNF formula and $\tau(\phi)$ be the respective AF. The following are equivalent properties respecting ϕ:

- The formula ϕ is satisfiable.
- The AF $\tau(\phi)$ has a stable extension.
- The AF $\tau(\phi)$ has a *non-empty* preferred extension.
- The argument ϕ in $\tau(\phi)$ is credulously accepted w.r.t. admissible semantics.

Satisfiability

Definition 7

Given a boolean formula ϕ, ϕ is said to be **satisfiable** iff there exists some assignment of variables within ϕ such that ϕ itself evaluates to true; if this is not the case, ϕ is said to be **unsatisfiable**.

Satisfiability of a 3-CNF formula can be quite predictable according to the ratio of the number of clauses, m, to the number of variables, n, which we will refer to as r.

Satisfiability Threshold

Fact 3

Let ϕ be a randomly drawn formula from the space of n variable, m clause 3-CNF formula. There exists constants $\langle \theta_k^l, \theta_k^u \rangle \in \mathbb{R}^+$ (with $\theta_k^l \leq \theta_k^u$) such that
Letting $r = m/n$,

$$\lim_{n \to \infty} \Pr[\phi \text{ is satisfiable}] = \begin{cases} 1, & \text{if } r < \theta_k^l \\ 0, & \text{if } r > \theta_k^u \end{cases}$$

Satisfiability Threshold

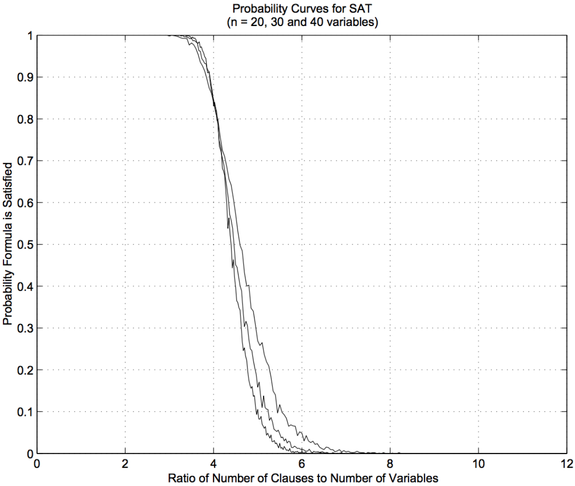


Figure 2.2: Probability curves for 3-SAT for $n = 20, 30, 40$ variables

Bailey, Delbert. 3-CNF Phase Transition. 2004. Phase Transitions Of Boolean Satisfiability Variants.

Question

Question?

"Is the pattern whereby random 3-CNF with a small value of *r* are almost certainly satisfiable whilst those with many clauses are not, reflected in the spectral properties of the AF defined through τ"?

Experimental Procedure

1. Set n, the number of variables and set m the number of clauses.

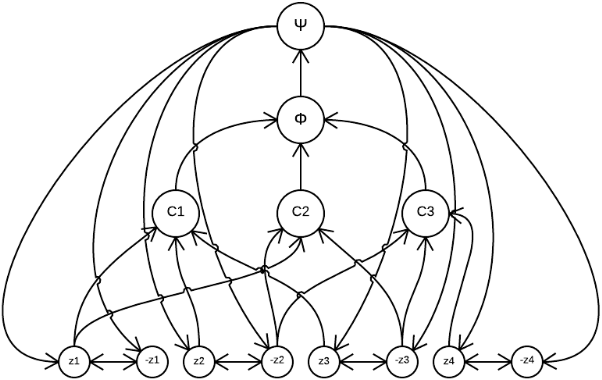
n	m	r
4	3	0.75

2. Generate a random n variable, m clause, 3-CNF formula, ϕ.

$$\phi = (z_1 \vee z_2 \vee z_3) \wedge (z_1 \vee \neg z_2 \vee \neg z_3) \wedge (\neg z_2 \vee \neg z_3 \vee z_4)$$

Experimental Procedure

3. Form the AF, $\tau(\phi)$.



Experimental Procedure

4. Form the adjacency matrix, $M^{\tau(\phi)}$.

$$
\begin{array}{c} \\ \psi \\ \phi \\ C_1 \\ C_2 \\ C_3 \\ z_1 \\ \neg z_1 \\ z_2 \\ \neg z_2 \\ z_3 \\ \neg z_3 \\ z_4 \\ \neg z_4 \end{array}
\begin{array}{c}
\begin{array}{ccccccccccccc} \psi & \phi & C_1 & C_2 & C_3 & z_1 & \neg z_1 & z_2 & \neg z_2 & z_3 & \neg z_3 & z_4 & \neg z_4 \end{array} \\
\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
\end{array}
$$

Experimental Procedure

5. Determine for the adjacency matrix $M^{\tau(\phi)}$,
   (a) The dominant eigenvalue λ_1.
   (b) The second largest eigenvalue λ_2.
   (c) The Estrada Index $\mathbf{EE}(\tau(\phi))$.

For this we used an eigenvalue calculator called JAMA, which is a package that provides fundamental operations of linear algebra.

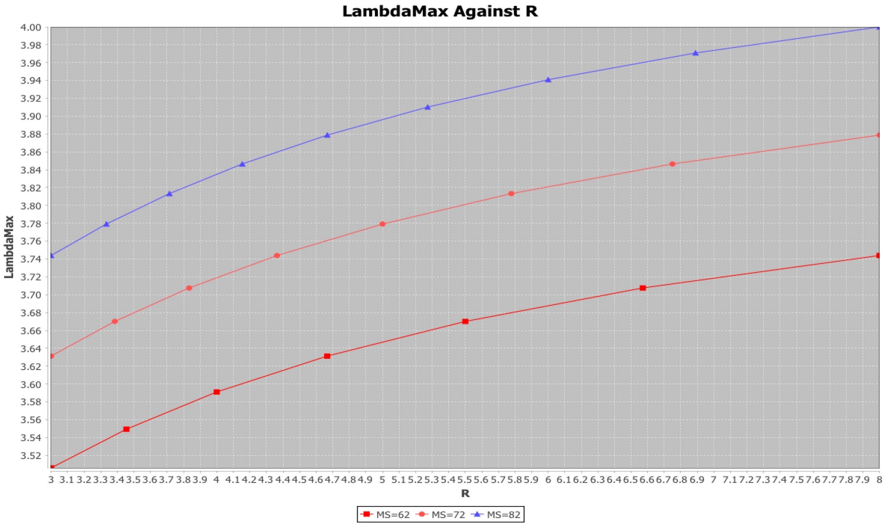
Other eigenvalues were considered such as λ_n, λ_{n-1} and $\lambda_{n/2}$, however these values were minute and provided no interesting insights.

Overview

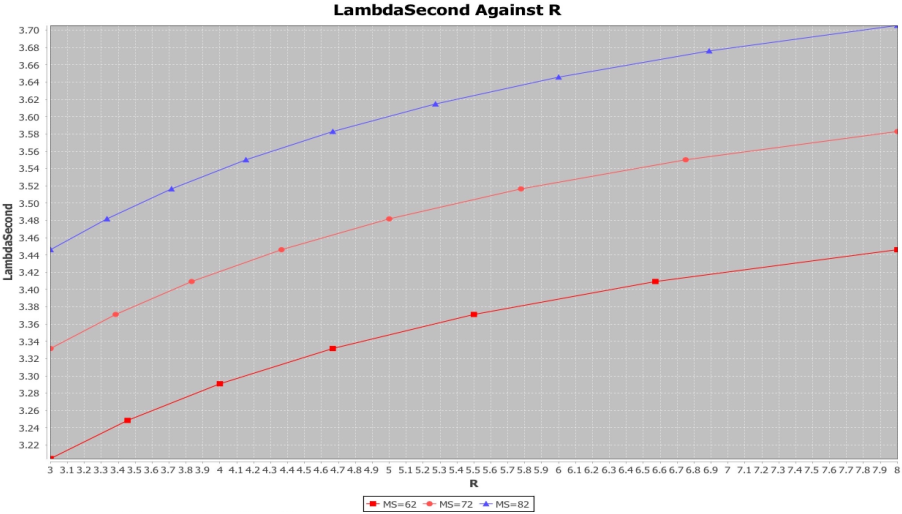
Results - λ_1 against R

NumClauses	NumVars	R	LambdaMax	MatrixSize
36	12	3	3.5060127607816627	62
38	11	3.45454545454545	3.5494037865195067	62
40	10	4	3.5911274782870497	62
42	9	4.66666666666667	3.6313263256695034	62
44	8	5.5	3.6701245557563675	62
46	7	6.57142857142857	3.7076311847742875	62
48	6	8	3.743942451076405	62

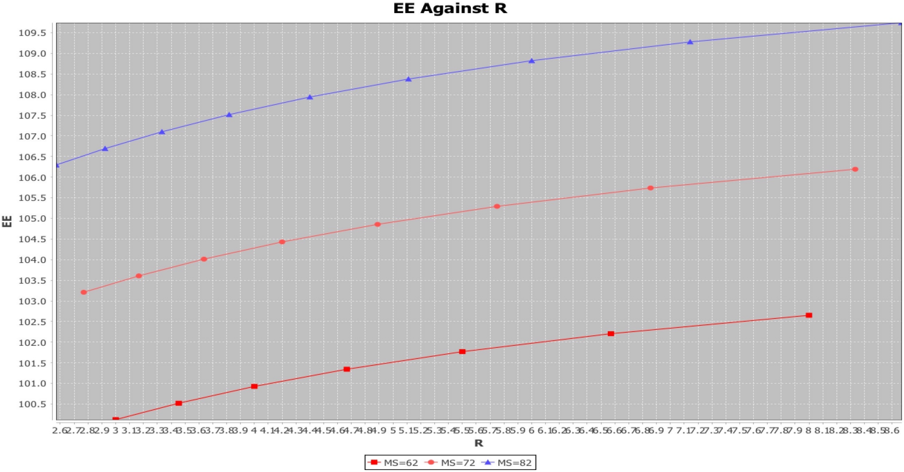
Results - λ_1 against R



Results - λ_2 against R



Results - *EE* against *R*



Overview

Conclusion

- It has been shown in the case of the AFs considered here that there exists a positive correlation between **R** and λ_1, λ_2 and **EE** in turn.
- Through the phase transition of boolean satisfiability, this means that AFs of this kind with spectra below certain thresholds will almost always be satisfiable (have a stable extension) and those with spectra higher than certain thresholds will almost always be unsatisfiable (does not have a stable extension).

Conclusion

- However, these argumentation frameworks are not typical frameworks.
- The purpose of this study was to establish some link between graph spectra and Dungian abstract argumentation semantics and to encourage further investigation into this area.

Future Work

1. Construct AFs with varying dominant eigenvalues (non-trivial task) and consider their semantic properties.
2. Analyse the spectral properties of the extension characterisations (Xu and Cayrol 2015).
3. Consider the spectral properties of real world AFs.
4. Consider **Laplacian** matrices as a representation.
5. Consider correlation with **eigenvectors**.

Questions

Thank you for listening

Any Questions?